18.05 Lecture 28
April 20, 2005

Review for Exam 2

pg. 280, Problem 5
$\mu = 300, \sigma = 10; X_1, X_2, X_3 \sim N(300, 100 = \sigma^2)$
$\mathbb{P}(X_1 > 290 \cup X_2 > 290 \cup X_3 > 290) = 1 - \mathbb{P}(X_1 \leq 290)\mathbb{P}(X_2 \leq 290)\mathbb{P}(X_3 \leq 290)$

$$= 1 - \mathbb{P}(\frac{x_1 - 300}{10} \leq -1)\mathbb{P}(\frac{x_2 - 300}{10} \leq -1)\mathbb{P}(\frac{x_3 - 300}{10} \leq -1)$$

Table for x = 1 gives 0.8413, x = -1 is therefore 1 - 0.8413 = 0.1587
$= 1 - (0.1587)^3 = 0.996$

pg. 291, Problem 11
600 seniors, a third bring both parents, a third bring 1 parent, a third bring no parents.
Find $\mathbb{P}(< 650$ parents)
$X_i - 0, 1, 2 \rightarrow$ parents for the ith student.
$\mathbb{P}(X_i = 2) = \mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 0) = \frac{1}{3}$
$\mathbb{P}(X_1 + ... + X_{600} < 650)$ - use central limit theorem.
$\mu = 0(1/3) + 1(1/3) + 2(1/3) = 1$
$\mathbb{E}X^2 = 0^2(1/3) + 1^2(1/3) + 2^2(1/3) = \frac{5}{3}$
$\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{5}{3} - 1 = \frac{2}{3}$
$\sigma = \sqrt{2/3}$

$$\mathbb{P}(\frac{\sqrt{600}(\frac{\sum x_i}{600} - 1)}{\sqrt{2/3}} < \frac{(\frac{650}{600} - 1)\sqrt{600}}{\sqrt{2/3}})$$

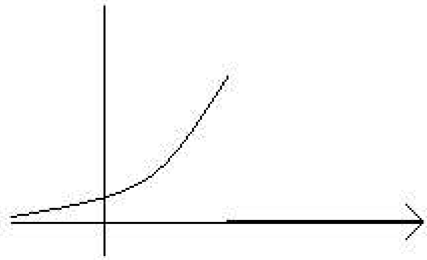
$$\mathbb{P}(\frac{\sqrt{n}(\overline{x} - \mu)}{\sigma} < 2.5) \approx N(0, 1), \mathbb{P}(Z \leq 2.5) = \phi(2.5) = 0.9938$$

pg. 354, Problem 10
Time to serve $X \sim E(\theta), n = 20, X_1, ..., X_{20}, \overline{x} = 3.8$ min
Prior distribution of θ is a Gamma dist. with mean 0.2 and std. dev. 1
$\alpha/\beta = 0.2, \alpha/\beta^2 = 1 \rightarrow \beta = 0.2, \alpha = 0.04$
Get the posterior distribution:
$f(x|\theta) = \theta e^{-\theta x}, f(x_1, ..., x_n|\theta) = \theta^n e^{-\theta \sum x_i}$
$f(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, f(\theta|x_1, ..., x_n) \sim \theta^{(\alpha+n)-1}e^{-(\beta+\sum x_i)\theta}$
Posterior is $\Gamma(\alpha + n, \beta + \sum x_i) = \Gamma(0.04 + 20, 0.2 + 3.8(20))$
Bayes estimator = mean of posterior distribution =

$$= \frac{20.04}{3.8(20) + 0.2}$$

Problem 4
$f(x|\theta) = \{e^{\theta - x}, x \geq 0; 0, x < 0\}$
Find the MLE of θ
Likelihood $\phi(\theta) = f(x_1|\theta) \times ... \times f(x_n|\theta)$
$= e^{\theta - x_1}...e^{\theta - x_n}I(x_1 \geq \theta, ..., x_n \geq \theta) = e^{n\theta - \sum x_i}I(\min(x_1, ..., x_n) \geq \theta)$

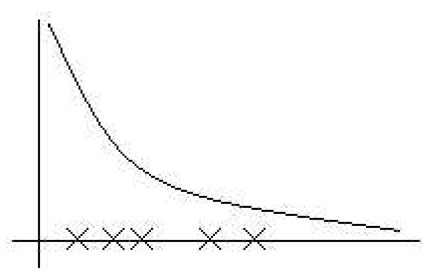
Maximize over θ.



Note that the graph increases in θ, but θ must be less than the min value.
If greater, the value drops to zero.
Therefore:

$$\hat{\theta} = \min(x_1, ..., x_n)$$

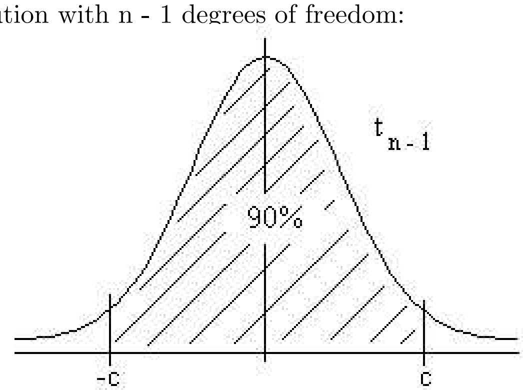
Also, by observing the original distribution, the maximum probability is at the smallest X_i.



p. 415, Problem 7:
To get the confidence interval, compute the average and sample variances:
Confidence interval for μ:

$$\overline{x} - c\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)} \leq \mu \leq \overline{x} - c\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}$$

To find c, use the t distribution with n - 1 degrees of freedom:



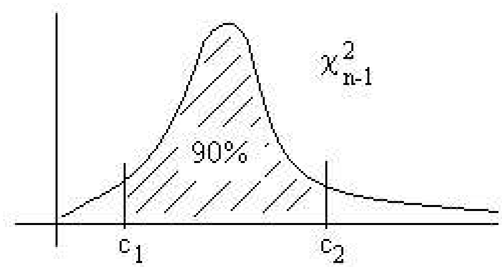
$t_{n-1} = t_{19}(-\infty, c) = 0.95, c = 1.729$

Confidence interval for σ^2:

$$\frac{\sqrt{n}(\overline{x} - \mu)}{\sigma} \sim N(0, 1), \frac{n(\overline{x^2} - (\overline{x})^2)}{\sigma^2} \sim \chi^2_{n-1}$$

$$t_{n-1} \sim \frac{N(0,1)}{\sqrt{\frac{1}{n-1}\chi^2_{n-1}}} = \frac{\sqrt{n}(\overline{x}-\mu)/\sigma}{\sqrt{\frac{1}{n-1}\frac{n(\overline{x^2}-(\overline{x})^2)}{\sigma^2}}} \sim t_{n-1}$$

Use the table for χ^2_{n-1}



$$c_1 \le \frac{n(\overline{x^2}-(\overline{x})^2)}{\sigma^2} \le c_2$$

From the Practice Problems:
(see solutions for more detail)

p. 196, Number 9
$\mathbb{P}(X_1 = defective) = p$
Find $\mathbb{E}(X - Y)$
$X_i = \{1, defective; -1, notdefective\}; X - Y = X_1 + ... + X_n$
$\mathbb{E}(X-Y) = \mathbb{E}X_1 + ... + \mathbb{E}X_n = n\mathbb{E}X_1 = n(1 \times p - 1(1-P)) = n(2p-1)$

p. 396, Number 10
$X_1, ..., X_6 \sim N(0,1)$
$c((X_1 + X_2 + X_3)^2 + (X_4 + X_5 + X_6)^2) \sim \chi^2_n$
$(\sqrt{c}(X_1 + X_2 + X_3))^2 + (\sqrt{c}(X_4 + X_5 + X_6))^2 \sim \chi^2_2$
But each needs a distribution of N(0, 1)
$\mathbb{E}\sqrt{c}(X_1 + X_2 + X_3) = \sqrt{c}(\mathbb{E}X_1 + \mathbb{E}X_2 + \mathbb{E}X_3) = 0$
$\mathrm{Var}(\sqrt{c}(X_1 + X_2 + X_3)) = c(\mathrm{Var}(x_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3)) = 3c$
In order to have the standard normal distribution, variance must equal 1.
$3c = 1, c = 1/3$

** End of Lecture 28